7.3

12) Let $f:[0,3]\to\mathbb{R}$ be defined by $f(x)=\begin{cases} x, & 0\le x<1 \\ 1, & 1\le x<2 \\ x, & 2\le x\le 3\end{cases}$

Obtain formulas for $F(x):=\int_0^x f$ and

Sketch the graphs of f and F. Where is F differentiable?
Evaluate at $F'(x)$ at all such points.

Pf: On $0\le x<1$,

$$F(x)=\int_0^x f(y)\,dy=\int_0^x y\,dy=\frac{1}{2}y^2\Big|_0^x=\frac{1}{2}x^2$$

On $1\le x<2$,

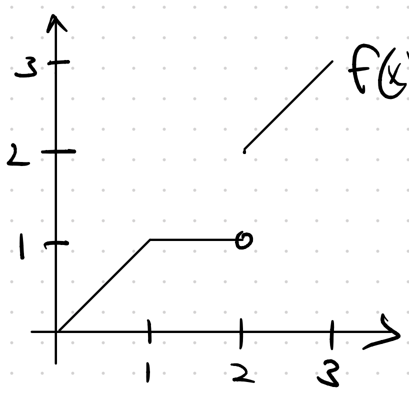
$$F(x)=\int_0^x f(y)\,dy=\int_0^1 f(y)\,dy+\int_1^x f(y)\,dy=\frac{1}{2}+\int_1^x dy$$

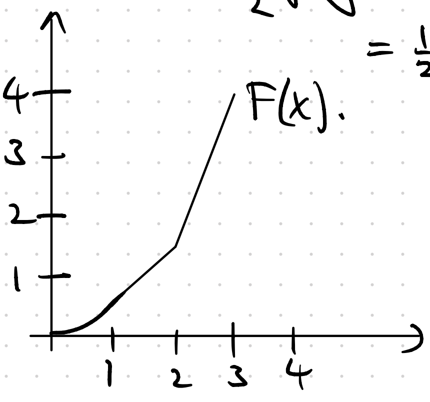
$$=\frac{1}{2}+(x-1)=x-\frac{1}{2}.$$

On $2\le x\le 3$

$$F(x)=\int_0^x f(y)\,dy=\frac{1}{2}+\left(2-\frac{1}{2}\right)-\left(-\frac{1}{2}\right)+\int_2^x y\,dy=\frac{5}{2}+\left(\frac{1}{2}x^2-2\right)$$

$$=\frac{1}{2}x^2-\frac{1}{2}$$





Note that for $x=1$, $\lim_{x\to 1^-} F'(x) = \lim_{x\to 1^-} x = 1$.

$\lim_{x\to 1^+} F'(x) = \lim_{x\to 1^+} 1 = 1$.

So $F'(1)$ does exist.

But at $x=2$,

$\lim_{x\to 2^-} F'(x) = \lim_{x\to 2} 1 = 1$.

$\lim_{x\to 2^+} F'(x) = \lim_{x\to 2} x = 2$ So $F'(2)$ DNE.

So $F'(x) = f(x)$ for $x \in [0,3] \setminus \{2\}$.

14) Show there does not exist a continuously differentiable function f on $[0,2]$ such that $f(0)=-1$, $f(2)=4$, $f'(x)\leq 2$ for $0\leq x\leq 2$.

Pf: By FTC, we have

$$f(x)-f(0)=\int_0^x f'(x)\,dx \leq \int_0^x 2\,dx = 2x.$$

So $f(x)\leq 2x+f(0) = 2x-1$ for $0\leq x\leq 2$.

Then $f(2)\leq 2\cdot 2-1 = 3 < 4$. So $f(2)=4$ is impossible. //.

7) Proof of Substitution Theorem.

Pf: Following the hint, we define

$$F(u) := \int_{\varphi(\alpha)}^{u} f(x)\,dx, \text{ for } u \in I.$$

$$H(t) := F(\varphi(t)) \text{ for } t \in J.$$

Then by the fundamental theorem of calculus,

$F'(u) = f(u)$ for $u \in I$, and so by the chain rule,

$H'(t) = F'(\varphi(t))\varphi'(t) = f(\varphi(t))\varphi'(t)$ and we hence that

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx \overset{FTC}{=} F(\varphi(\beta)) - F(\varphi(\alpha))$$

$$= H(\beta) - H(\alpha)$$

$$\overset{FTC}{=} \int_{\alpha}^{\beta} H'(t)\,dt$$

$$= \int_{\alpha}^{\beta} f(\varphi(t))\varphi'(t)\,dt.$$

as required. //

21) Let $f, g \in \mathcal{R}[a,b]$

a) If $t \in \mathbb{R}$, show that $\int_a^b (tf \pm g)^2 \geq 0$

b) Use (a) to show that $2\left|\int_a^b fg\right| \leq t\int_a^b f^2 + \left(\frac{1}{t}\right)\int_a^b g^2$ for $t > 0$

(c) If $\int_a^b f^2 = 0$, show that $\int_a^b fg = 0$

(d) Now prove that $\left|\int_a^b fg\right|^2 \leq \left(\int_a^b |fg|\right)^2 \leq \left(\int_a^b f^2\right)\cdot\left(\int_a^b g^2\right)$,

the Cauchy-Schwarz inequality.

Pf: (a) by the Product Theorem (Thm 7.3.16), $tf, g \in \mathcal{R}[a,b]$, then the Composition Theorem (Thm 7.3.14) implies that $(tf \pm g)^2 \in \mathcal{R}[a,b]$. Finally, since $(tf \pm g)^2 \geq 0$, $\int_a^b (tf \pm g)^2 \geq 0$.

(b) Expanding out (a), we get

$$\int_a^b t^2f^2 + 2tfg + g^2 \geq 0 \Rightarrow -2t\int_a^b fg \leq t^2\int_a^b f^2 + \int_a^b g^2$$

$$\int_a^b t^2f^2 - 2tfg + g^2 \geq 0 \Rightarrow 2t\int_a^b fg \leq t^2\int_a^b f^2 + \int_a^b g^2.$$

So dividing by t ($t > 0$) and taking absolute value we obtain

$$2\left|\int_a^b fg\right| \leq t\int_a^b f^2 + \left(\frac{1}{t}\right)\int_a^b g^2.$$

(c) if $\int_a^b f^2 = 0$, (b) above becomes

$$2\left|\int_a^b fg\right| \leq \left(\frac{1}{t}\right)\int_a^b g^2 \quad \text{for all } t > 0.$$

So letting $t \to \infty$ yields

$$2\left|\int_a^b fg\right| \leq 0 \implies \int_a^b fg = 0.$$

(d) Note by (c) above, if $\int_a^b f^2 = 0$, (d) is trivially the inequality $0 \leq 0$.

If $\int_a^b f^2 \neq 0$, then let $t = \left(\int_a^b g^2 \Big/ \int_a^b f^2\right)^{1/2}$ in (b) above

and, replacing f by $|f|$ and g by $|g|$ in (b), we get,

Cor 7.3.15

$$2\left|\int_a^b fg\right| \overset{\downarrow}{\leq} 2\int_a^b |fg| \leq \left(\int_a^b g^2 \Big/ \int_a^b f^2\right)^{1/2} \int_a^b f^2 + \left(\int_a^b f^2 \Big/ \int_a^b g^2\right)^{1/2} \int_a^b g^2$$

$$= 2\left(\int_a^b g^2\right)^{1/2}\left(\int_a^b f^2\right)^{1/2}$$

$$\implies \left|\int_a^b fg\right|^2 \leq \left(\int_a^b |fg|\right)^2 \leq \left(\int_a^b f^2\right)^{1/2}\left(\int_a^b g^2\right)^{1/2} \quad \text{as required} \;//$$

(since $|f|^2 = f^2$, $|g|^2 = g^2$).

7.4

3) Let f and g be bounded functions on $I := [a,b]$. If $f(x) \leq g(x)$ for all $x \in I$, show that $L(f) \leq L(g)$ and $U(f) \leq U(g)$.

Pf: let P be a partition of $[a,b]$ with subintervals $I_k := [x_{k-1}, x_k]$.

Then since $f(x) \leq g(x)$ for all $x \in I$,

$$m_k(f) = \inf\{f(x) : x \in I_k\} \leq \inf\{g(x) : x \in I_k\} = m_k(g),$$

so $L(f;P) \leq L(g;P)$ for the partition P.

since P was chosen arbitrarily, this implies

$$L(f) \leq L(g).$$

Similarly, $U(f) \leq U(g)$. //

8) Let f be continuous on $I := [a,b]$ and assume $f(x) \geq 0$ for all $x \in I$. Prove that if $L(f) = 0$, then $f(x) = 0$ for all $x \in I$.

Pf: Suppose for contradiction that $\exists c \in [a,b]$ s.t. $f(c) > 0$. Then by continuity of f, there is a small nbhd V_δ of c s.t. $f(x) > \frac{f(c)}{2} > 0$ on $V_\delta \cap [a,b]$. Then for some partition P, we have $L(f;P) > 0$, and hence $L(f) > 0$, which is a contradiction //.